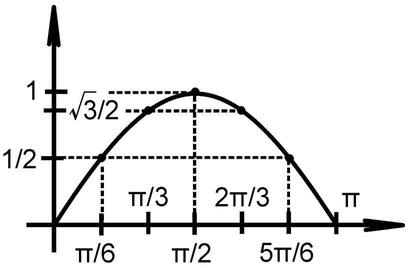
Analize III, pismeni ispit, 22.09.2014.



1. Dio grafika f-je $y = f(x)$ je prikazan na slici lijevo. Datu funkciju pretvoriti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2}$.

2. Odrediti zapreminu tijela ograničeno površima $z = x^2 + y^2$ i $2z = 1 - x^2 - y^2$.

3. (20%)(a) Primjenom formule Gauss-Ostrogradskog izračunati površinski integral druge vrste $\oiint\limits_{-S} z\mathrm{d}x\mathrm{d}y + y\mathrm{d}x\mathrm{d}z + x\mathrm{d}y\mathrm{d}z$ gdje je S-površina kocke ograničena ravnima $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

(80%)(b) Izračunati bez korištenja formule Gauss-Ostrogradskog integral

$$\oiint\limits_{S} x\mathrm{d}y\mathrm{d}z + (x+y)\mathrm{d}z\mathrm{d}x + (x+y+z)\mathrm{d}x\mathrm{d}y$$

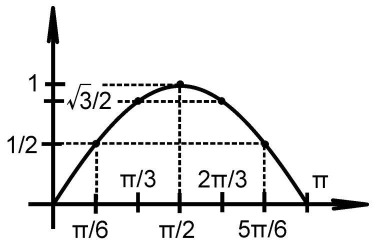
gdje je S spoljna strana površi $x^2 + y^2 + z^2 = a^2$ ($a > 0$).

4. Zadano je vektorko polje $\vec{f} = x\vec{i} + (e^y \sin z)\vec{j} + (e^y \cos z)\vec{k}$.

(20%)(a) Dokažite da je polje potencijalno. (80%)(b) Izračunajte potencijal polja $\vec{f}$.

VAŽNO: Ovaj papir treba predati zajedno s rješenjima zadataka! Prije rješenja prepisati postavku (tekst) zadatka. Ispit pisati isključivo hemijskom olovkom plave ili crne tinte.

Analize III, pismeni ispit, 22.09.2014.



1. Dio grafika f-je $y = f(x)$ je prikazan na slici lijevo. Datu funkciju pretvoriti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2}$.

2. Odrediti zapreminu tijela ograničeno površima $z = x^2 + y^2$ i $2z = 1 - x^2 - y^2$.

3. (20%)(a) Primjenom formule Gauss-Ostrogradskog izračunati površinski integral druge vrste $\oiint\limits_{-S} z\mathrm{d}x\mathrm{d}y + y\mathrm{d}x\mathrm{d}z + x\mathrm{d}y\mathrm{d}z$ gdje je S-površina kocke ograničena ravnima $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

(80%)(b) Izračunati bez korištenja formule Gauss-Ostrogradskog integral

$$\oiint\limits_{S} x\mathrm{d}y\mathrm{d}z + (x+y)\mathrm{d}z\mathrm{d}x + (x+y+z)\mathrm{d}x\mathrm{d}y$$

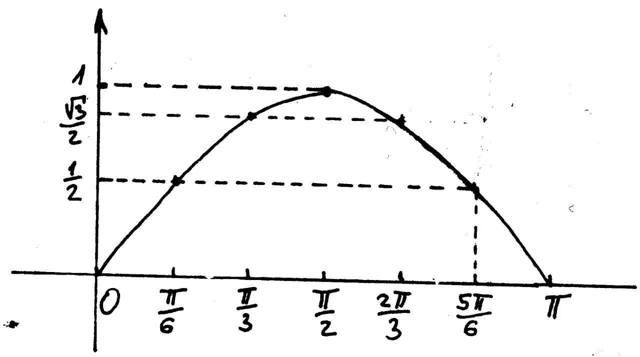
gdje je S spoljna strana površi $x^2 + y^2 + z^2 = a^2$ ($a > 0$).

4. Zadano je vektorko polje $\vec{f} = x\vec{i} + (e^y \sin z)\vec{j} + (e^y \cos z)\vec{k}$.

(20%)(a) Dokažite da je polje potencijalno. (80%)(b) Izračunajte potencijal polja $\vec{f}$.

VAŽNO: Ovaj papir treba predati zajedno s rješenjima zadataka! Prije rješenja prepisati postavku (tekst) zadatka. Ispit pisati isključivo hemijskom olovkom plave ili crne tinte.

(#) Dio grafika f-je $y=f(x)$ je prikazan na slici.



Datu f-ju pretvoriti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2}$.

Rj. Furijerov red za f-ju $y=f(x)$ na intervalu (a,b) glasi

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right) \quad \ldots(1)$$

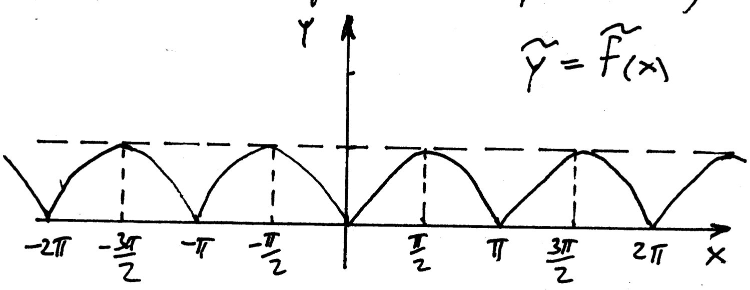
gdje se Furijer-ovi koeficijenti računaju po formuli

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx, \quad a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx, \quad b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx \quad \ldots(2)$$

Prema formuli (1) da bi f-ju pretvorili u Furijer-ov red samo po cos-inusima, potrebno je i dovoljno imati f-ju za koju će vrijediti da je $b_n=0$. Prema formuli (2) da bi b_n bio jednak nuli, interval (a,b) mora biti simetričan u odnosu na nulu i f-ja $f(x)$ mora biti parna (zato što

$$b_n = \frac{2}{b-a}\int_a^b \underbrace{\underbrace{f(x)}_{\text{parna}} \underbrace{\sin\frac{2n\pi x}{b-a}}_{\text{neparna}}}_{\text{neparna}}\,dx \;).$$

Prema tome pravimo proširenje date f-je:



Naravno f-ja koju pretvaramo u Furijer-ov red mora biti periodična.

Prvo primjetimo da je data f-ja $y=f(x)$ u stvari f-ja $y=\sin x$ na intervalu $(0,\pi)$. Proširenje date f-je je u stvari f-ja $\tilde{y}=|\sin x|$.

$(a,b)=(-\pi,\pi)$, $\quad \frac{2}{b-a}=\frac{2}{\pi-(-\pi)}=\frac{2}{2\pi}=\frac{1}{\pi}$, $\quad \frac{2n\pi x}{b-a}=\frac{2n\pi x}{2\pi}=nx$

F-ja $\tilde{y}=|\sin x|$ je parna $\Rightarrow$ $b_n=0$.

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\tilde{f}(x)\cos nx}_{\text{parna f-ja}}\,dx=\frac{2}{\pi}\int_0^{\pi}\tilde{f}(x)\cos nx\,dx=$$

$$=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\left(\sin(n+1)x+\sin(n-1)x\right)dx=$$

$$\begin{aligned}&\sin(A+B)=\sin A\cos B+\sin B\cos A\\+\;&\sin(A-B)=\sin A\cos B-\sin B\cos A\\\hline&\sin A\cos B=\tfrac{1}{2}\left(\sin(A+B)+\sin(A-B)\right)\end{aligned}$$

$$=\frac{1}{\pi}\left(\frac{-1}{1+n}\cos(1+n)x\Big|_0^{\pi}+\frac{-1}{1-n}\cos(1-n)x\Big|_0^{\pi}\right)=$$

$$=\frac{1}{\pi}\left(\frac{-1}{1+n}\Big(\underbrace{\cos(1+n)\pi}_{(-1)^{n+1}}-1\Big)+\frac{-1}{1-n}\Big(\underbrace{\cos(1-n)\pi}_{(-1)^{n-1}}-1\Big)\right)=$$

$$=\frac{1}{\pi}\left(\frac{(-1)^n+1}{1+n}+\frac{(-1)^n+1}{1-n}\right)=\frac{1}{\pi}\,\frac{((-1)^n+1)(1+n+1-n)}{(1+n)(1-n)}=\frac{1}{\pi}\,\frac{2}{(1+n)(1-n)}\left((-1)^n+1\right)$$

za $n=1,2,3,\dots$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\tilde{f}(x)\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x\,dx=\frac{2}{\pi}\left(-\cos x\Big|_0^{\pi}\right)=\frac{-2}{\pi}(-1-1)=\frac{4}{\pi}$$

$$\tilde{f}(x)\sim\frac{4}{\pi}+\sum_{n=1}^{\infty}\frac{2}{\pi}\,\frac{(-1)^n+1}{1-n^2}\cos nx=\frac{4}{\pi}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{1-4k^2}$$

Prema tome $\quad \sin x\sim\frac{4}{\pi}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{1-4k^2}$, $\quad x\in(0,\pi)$

Za $x=\frac{\pi}{2}$ imamo:

$$\sin\frac{\pi}{2}=\frac{4}{\pi}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2k\cdot\frac{\pi}{2}}{1-4k^2}\Rightarrow 1-\frac{4}{\pi}=\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos k\pi}{1-4k^2}\quad /\cdot\pi$$

$$4\sum_{k=1}^{\infty}\frac{(-1)^k}{1-4k^2}=\pi-4$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}=\frac{\pi}{4}-1\quad \text{tražena suma}$$

(#) Odrediti zapreminu tijela ograničeno površima

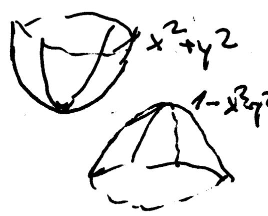
$$z = x^2 + y^2 \quad i \quad 2z = 1 - x^2 - y^2.$$

Rj. Zapreminu tijela računamo po formuli $V = \iiint\limits_{\Omega} dx\,dy\,dz$.

Primjetimo da je površ $z = x^2 + y^2$ uvijek iznad xOy ravni $(z \geq 0)$, dok za površ $z = \frac{1}{2}(1 - x^2 - y^2)$ uvijek vrijedi $z \leq \frac{1}{2}$.

Pitanje: Kako odrediti ortogonalnu projekciju presjeka dvije date površi na xOy ravan?

Ortogonalnu projekciju ćemo dobiti ako se iz sistema $z = x^2 + y^2$, $z = \frac{1}{2}(1 - x^2 - y^2)$ na neki način riješimo z-ta.



$$\begin{aligned} z &= x^2 + y^2 \\ z &= \frac{1}{2}(1 - x^2 - y^2) \\ \hline x^2 + y^2 &= \frac{1}{2}(1 - x^2 - y^2) \end{aligned}$$

$$2x^2 + 2y^2 = 1 - x^2 - y^2$$

$$x^2 + y^2 = \frac{1}{3}$$

ortogonalna projekcija presjeka dvije površi

za xOy ravan uvedimo polarne koordinate tj. za trostruki integral uvedimo cilindrične koordinate.

$$V = \iiint\limits_{\Omega} dx\,dy\,dz = \iint\limits_{D} dx\,dy \int\limits_{x^2+y^2}^{\frac{1}{2}(1-x^2-y^2)} dz = \iint\limits_{D} \left[\left(\frac{1}{2} - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) - \left(x^2 + y^2\right)\right] dx\,dy =$$

$$= \iint\limits_{D} \underbrace{\left(\frac{1}{2} - \frac{3}{2}x^2 - \frac{3}{2}y^2\right)}_{\frac{1}{2} - \frac{3}{2}(x^2+y^2)} dx\,dy = \left| \begin{array}{l} \text{polarne koordinate} \\ x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ dx\,dy = \rho\,d\rho\,d\varphi \\ x^2 + y^2 = \rho^2 \end{array} \quad D \xrightarrow{\text{transformiše}} D' : \begin{cases} 0 \leq \rho \leq \frac{\sqrt{3}}{3} \\ 0 \leq \varphi \leq 2\pi \end{cases} \right| =$$

$$= \iint\limits_{D'} \left(\frac{1}{2} - \frac{3}{2}\rho^2\right)\rho\,d\rho\,d\varphi = \int\limits_{0}^{2\pi} d\varphi \int\limits_{0}^{\frac{\sqrt{3}}{3}} \left(\frac{1}{2}\rho - \frac{3}{2}\rho^3\right) d\rho = \ldots = \frac{\pi}{12}$$

Primjerom formule Gauss-Ostrogradskog izračunati površinski integral druge vrste

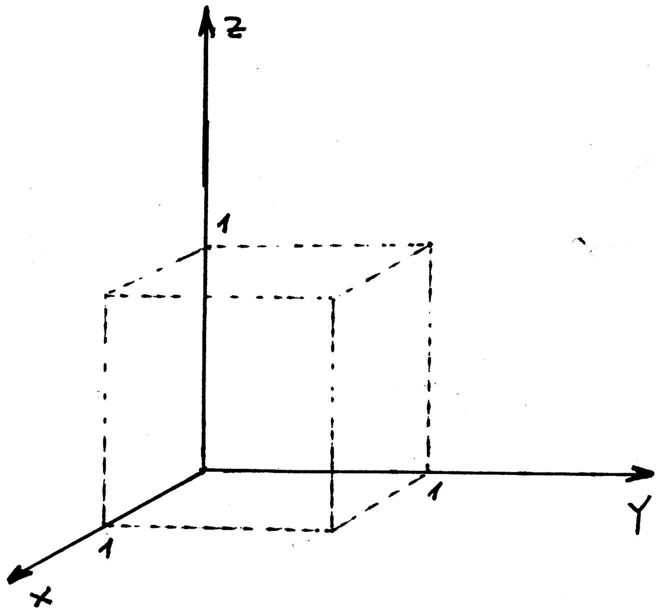
$$\oiint_{-\sigma} z\,dx\,dy + y\,dx\,dz + x\,dy\,dz$$

gdje je σ - površina kocke ograničene ravnima $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Rj.

Formula Gauss-Ostrogradskog površinski integral druge vrste svodi na trostruki integral

$$\underline{\oiint_{S} P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy} = \underline{\iiint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz}$$



$P(x,y,z) = x$

$Q(x,y,z) = y$

$R(x,y,z) = z$

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z} = 3$$

$$\oiint_{-\sigma} z\,dx\,dy + y\,dx\,dz + x\,dy\,dz \overset{\text{form. Gauss-Ostr.}}{=\!=}$$

$$= -3\underbrace{\iiint_{\Omega} dx\,dy\,dz}_{\text{zapremina kocke}} = -3 \quad \text{traženo rješenje}$$

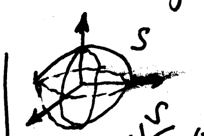
(#) Izračunati bez korištenja formule Gauss-Ostrogradskog integral

$$\iint_S x\,dy\,dz + (x+y)\,dz\,dx + (x+y+z)\,dx\,dy,$$

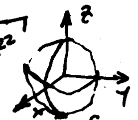
gdje je S spoljna strana površi $x^2+y^2+z^2=a^2$ $(a>0)$.

Rj. Dati integral podjelimo na tri dijela $I_1 = \iint_S x\,dy\,dz$, $I_2 = \iint_S (x+y)\,dz\,dx$ i $I_3 = \iint_S (x+y+z)\,dx\,dy$, pa izračunajmo svaki od ovih integrala posebno. Krenimo redom

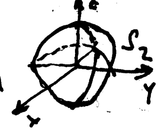
$\iint_S x\,dy\,dz =$ |



$x=\sqrt{a^2-y^2-z^2}$



površ S podjelimo na dva dijela S_1 i S_2



$x=-\sqrt{a^2-y^2-z^2}$

| $= \iint_{S_1} x\,dy\,dz + \iint_{S_2} x\,dy\,dz =$

$=$ | pravimo projekciju površi na yOz ravan
u oba slučaja $D: \{ y^2+z^2=a^2$
za vektor normale $\vec{n}$ na S_1 vrijedi $0\le\alpha\le\frac{\pi}{2}$
za vektor normale $\vec{n}$ na S_2 vrijedi $\frac{\pi}{2}<\alpha\le\pi$ | $= \iint_D \sqrt{a^2-y^2-z^2}\,dy\,dz - \iint_D \left(-\sqrt{a^2-y^2-z^2}\right) dy\,dz$

$=$ | uvedimo polarne koordinate
$y=\rho\cos\varphi$
$z=\rho\sin\varphi$
$dy\,dz=\rho\,d\varphi\,d\rho$
$D \xrightarrow{\text{transform.}} D': \begin{cases} 0\le\rho\le a \\ 0\le\varphi\le 2\pi \end{cases}$
$y^2+z^2=\rho^2$ | $= 2\iint_{D'} \sqrt{a^2-\rho^2}\,\rho\,d\rho\,d\varphi =$ | $d(a^2-\rho^2) = -2\rho\,d\rho$
$\rho\,d\rho = -\frac{1}{2}\,d(a^2-\rho^2)$

$$= 2\cdot\left(\frac{-1}{2}\right)\int_0^{2\pi} d\varphi \int_0^a (a^2-\rho^2)^{\frac{1}{2}}\,d(a^2-\rho^2) = (-1)\cdot 2\pi\cdot \underbrace{\frac{(a^2-\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{\rho=0}^{\rho=a}}_{\frac{2}{3}\left(0-(a^2)^{\frac{3}{2}}\right)} = \frac{4}{3}a^3\pi$$

$\iint_S (x+y)\,dz\,dx =$ | površ S podjelimo na dva dijela

$y=-\sqrt{a^2-x^2-z^2}$

$y=\sqrt{a^2-x^2-z^2}$

| $= \iint_{S_1} (x+y)\,dz\,dx + \iint_{S_2} (x+y)\,dz\,dx$

$$= \left| \begin{array}{l} \text{pravimo projekciju površi na } xOz \text{ ravan} \\ \text{u oba slučaja } E: x^2+z^2=a^2 \\ \text{za vektor normale } \vec{n} \text{ na } S_1 \text{ vrijedi } \frac{\pi}{2} \le \beta \le \pi \\ \text{za vektor normale } \vec{n} \text{ na } S_2 \text{ vrijedi } 0 \le \beta \le \frac{\pi}{2} \end{array} \right| =$$

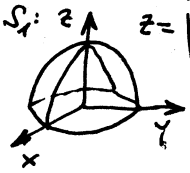
$$= -\iint_E \left(x - \sqrt{a^2-x^2-z^2}\right) dz\,dx + \iint_E \left(x + \sqrt{a^2-x^2-z^2}\right) dz\,dx =$$

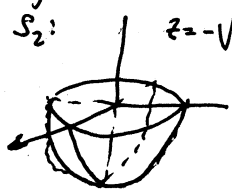
$$= 2\iint_E \sqrt{a^2-x^2-z^2}\, dz\,dx = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x=\rho\cos\varphi \\ z=\rho\sin\varphi \\ dx\,dz=\rho\,d\rho\,d\varphi \end{array} \quad E \xrightarrow{\text{transform.}} E': \begin{cases} 0 \le \rho \le a \\ 0 \le \varphi \le 2\pi \end{cases} \quad x^2+z^2=\rho^2 \right| =$$

$$= 2\iint_{E'} \sqrt{a^2-\rho^2}\,\rho\, d\rho\, d\varphi = \left| \begin{array}{l} d(a^2-\rho^2) = -2\rho\, d\rho \\ \rho\, d\rho = -\frac{1}{2} d(a^2-\rho^2) \end{array} \right| = 2\cdot\left(\frac{-1}{2}\right) \int_0^{2\pi} d\varphi \int_0^a (a^2-\rho^2)^{\frac{1}{2}}\, d(a^2-\rho^2)$$

$$= (-1)\cdot 2\pi \cdot \frac{2}{3} \underbrace{(a^2-\rho^2)^{\frac{3}{2}}\Big|_0^a}_{0-(a^2)^{\frac{3}{2}}} = \frac{4}{3}\pi a^3$$

$$\iint_S (x+y+z)\,dx\,dy = \left| \begin{array}{l} \text{površ } S \text{ podjelimo na dva dijela} \\ S_1: z=\sqrt{a^2-x^2-y^2} \qquad S_2: z=-\sqrt{a^2-x^2-y^2} \end{array} \right| = \iint_{S_1} (x+y+z)\,dx\,dy + \iint_{S_2} (x+y+z)\,dx\,dy$$





$$= \left| \begin{array}{l} \text{pravimo projekciju površi } S \text{ na } xOy \text{ ravan} \\ \text{u oba slučaja } F: x^2+y^2=a^2 \\ \text{za vektor normale } \vec{n} \text{ na } S_1 \text{ vrijedi } 0 \le \gamma \le \frac{\pi}{2} \\ \text{za vektor normale } \vec{n} \text{ na } S_2 \text{ vrijedi } \frac{\pi}{2} \le \gamma \le \pi \end{array} \right| = \iint_F \left(x+y+\sqrt{a^2-x^2-y^2}\right) dx\,dy - \iint_F \left[x+y-\sqrt{a^2-x^2-y^2}\right] dx\,dy$$

$$= 2\iint_F \sqrt{a^2-x^2-y^2}\, dx\,dy = \dots = \frac{4}{3}a^3\pi$$

Vrijednost datog integrala je $I = I_1 + I_2 + I_3 = 4a^3\pi$

Provjerimo rezultat upotrebom formule Gauss-Ostrogradskog

$$\iint_S x\,dy\,dz + (x+y)\,dz\,dx + (x+y+z)\,dx\,dy = \iiint_\Omega (1+1+1)\,dx\,dy\,dz = 3\cdot\frac{4}{3}a^3\pi = 4a^3\pi.$$

(#) Zadano je vektorsko polje $\vec{F} = x\vec{i} + (e^y \sin z)\vec{j} + (e^y \cos z)\vec{k}$.
Dokažite da je polje potencijalno i izračunajte njegov potencijal.

Rj.

Prisjetimo se:
Ako je $\operatorname{rot}\vec{v} = \vec{0}$ tada kažemo da je $\vec{v}$ potencijalno polje.

$$\operatorname{rot}\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

U našem slučaju imamo

$$\frac{\partial}{\partial y}(e^y \cos z) = e^y \cos z \qquad \frac{\partial}{\partial x}(e^y \cos z) = 0 \qquad \frac{\partial}{\partial x}(e^y \sin z) = 0$$

$$\frac{\partial}{\partial z}(e^y \sin z) = e^y \cos z \qquad \frac{\partial}{\partial z}(x) = 0 \qquad \frac{\partial}{\partial y}(x) = 0$$

Prema tome $\operatorname{rot}\vec{F} = (e^y \cos z - e^y \cos z,\ 0-0,\ 0-0) = (0,0,0) \Rightarrow$
$\Rightarrow$ polje $\vec{F}$ je potencijalno polje.

Prisjetimo se
Potencijalom polja $\vec{v}$ nazivamo f-ju u za koju vrijedi da je
$\vec{v} = \operatorname{grad} u$.

$$u = u(x,y,z) \Rightarrow du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz, \quad \operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\frac{\partial u}{\partial x} = x \Rightarrow u = \frac{1}{2}x^2 + \varphi(y,z)$$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= e^y \sin z & \frac{\partial u}{\partial y} &= \varphi'_y \\ \frac{\partial u}{\partial z} &= e^y \cos z & \frac{\partial u}{\partial z} &= \varphi'_z \end{aligned}\right\} \Rightarrow \begin{aligned} \varphi'_y &= e^y \sin z \Rightarrow \varphi = e^y \sin z + \psi(z) \\ \varphi'_z &= e^y \cos z \qquad \varphi'_z = e^y \cos z + \psi'_z \end{aligned}$$

$$\Downarrow$$

$$\psi'_z = 0 \Rightarrow \psi = C$$

$$\Rightarrow \varphi = e^y \sin z + C \Rightarrow u = \frac{1}{2}x^2 + e^y \sin z + C$$

Potencijal vektorskog polja $\vec{F}$ je $u = \frac{1}{2}x^2 + e^y \sin z + C$.